PENDULUM MODELS
FOR THE OSCILLATION OF CONFINED LIQUID

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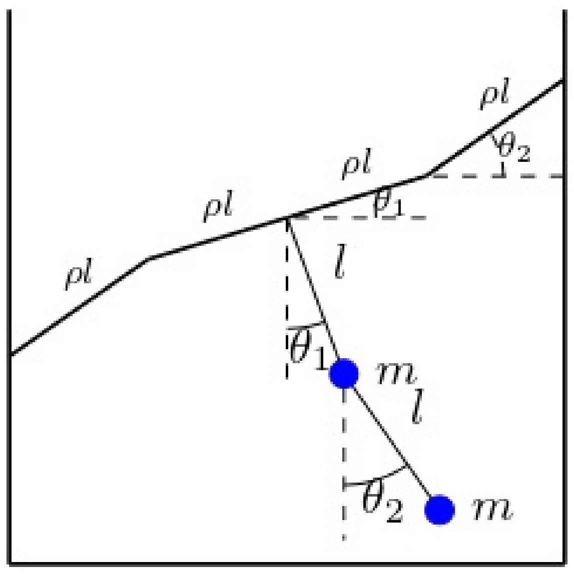
ABSTRACT

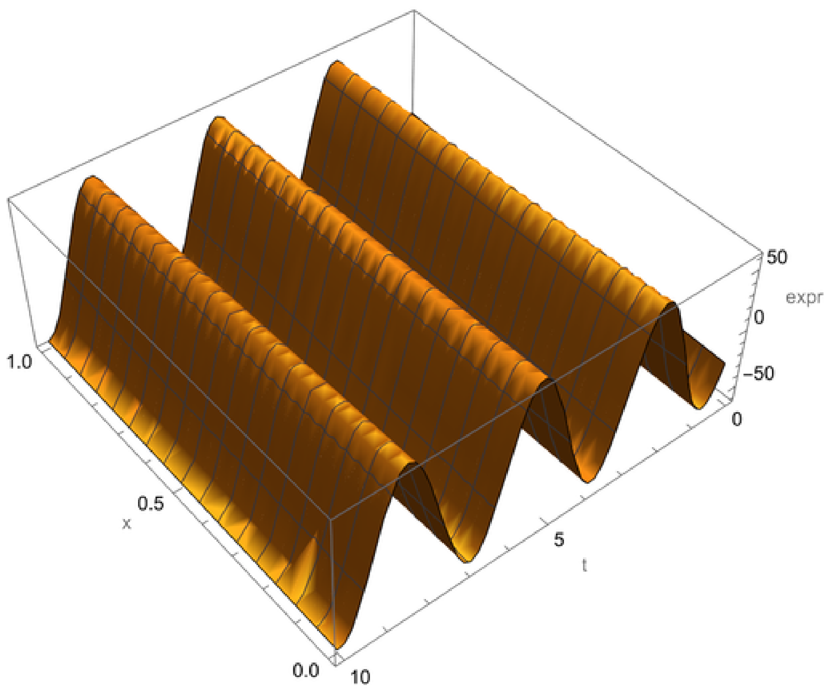
We are motivated by the paper "Modelling Transient Liquid Slosh Behavior at Variable Operating Speeds Induced by Intermittent Motions in Packaging Machines" which models liquid slosh using a single pendulum. We attempt to improve the model via a multi-pendulum model to more accurately model the surface behaviour of the oscillation of the liquid.

PROPOSED MODELS

DISECRETE MODEL



From the aforementioned paper, the surface of the liquid can be thought of as a plane which oscillates perpendicularly to an imaginary pendulum. We know from practice that the liquid's surface do not stay linear during this oscillation, which requires an adjustment, ideally through an interpolation of points. With this idea, we propose an n-pendulum model, each model the motion of 2 points on the liquid surface which is a rotation of 90 degrees clockwise and anticlockwise, interpolating the collection of all these points should give the surface behaviour. The Figure on the left is corresponding to proposed double pendulum model.



CONTINUOUS MODEL

Similar to the previous model, instead of providing estimates for discrete points on the surface of the liquid to then interpolate, we could create a continuous model based on vibrations of a string under an oscillating force. This gives the inhomogeneous wave equation at bottom left.

Define the differential equation:

$$\frac{\partial^2 u(x,t)}{\partial t^2} - c^2\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{F}{m}\cos(wt) - d\left(\frac{\partial u(x,t)}{\partial t}\right)^2$$

Define initial conditions:

$$u(x,0) = 0$$
$$\left.\frac{\partial u(x,t)}{\partial t}\right|_{t=0} = 0$$
$$u(0,t) = 0$$

SOLUTION

Through some approximations we have arrived at the form:

$$u(x,t) = \left(\frac{F}{m}\right)\left(\frac{c}{\omega}\right)\Big[-\sin(\omega(t-(l-x)/c)) - 4\sin(\omega t) + \sin(\omega(t-x/c)) + \sin(\omega(t+(l-x)/c)) + \sin(\omega(t+x/c)) + \frac{1}{2}\cos\left(\frac{\omega l}{c}\right)\left(-\sin(\omega(x/c-t)) + \sin(\omega(x/c+t))\right)\Big]$$

The figure on the left plots the wave described.

LAGRANGIAN FOR THE DISCRETE MODEL

We attempt to obtain a differential equation via the Lagrangian Formalism where we find the lagrangian to be:

$$\mathcal{L} = m\dot{x}^2 + m\ell\dot{x}(2\cos(\theta_1)\dot{\theta_1} + \cos(\theta_2)\dot{\theta_2}) + m\ell\theta_1\theta_2\cos(\theta_1-\theta_2) + \tfrac{1}{2}(2m\ell^2\dot{\theta_1}^2 + m\ell\dot{\theta_2}^2) + mg\ell(2\cos(\theta_1)+\cos(\theta_2)) + d\ell^2\cos^2(\theta_1)\dot{\theta_1}^2 + d\ell^2\cos(\theta_1)\dot{\theta_1}\cos(\theta_2)\dot{\theta_2} + \tfrac{1}{2}(d\ell^2\cos^2(\theta_2)\dot{\theta_2}^2 + \tfrac{1}{2}M\dot{x}.$$

Where the differential equation with respect to x is given by:

$$2m\ddot{x}(t) + M\ddot{x}(t) + m\ell\left(-2\sin(\theta_1(t))\dot{\theta_1}(t)^2 - \sin(\theta_2(t))\dot{\theta_2}(t)^2 + 2\cos(\theta_1(t))\ddot{\theta_1}(t) + \cos(\theta_2(t))\ddot{\theta_2}(t)\right) = 0$$

Since an analytic solution seems impractical for all degrees of freedoms we use numerical techniques (NDsolve in Mathematica and 4-stage Runge-Kutta in Python) to get a solution.

RESULTS & DISCUSSION

DISCRETE MODEL:

From our numerical approach, we have obtained several plots (in Mathematica, using NDsolve) for the n = 2 case, which diverges. This is likely due to the errors of approximations of the RK4 method which is exacerbated by the dependent terms in the coupled equation. This problem can be solved if we can find a way to separate the variables.

CONTINUOUS MODEL:

On the string model, we have been able to derive the wave equation (with damping term). Numerical approximation without damping (as the damping term will change the type of equation, making it non-standard. We can attempt to solve it via the Sturm-Liouville method which gives the wave plot on the left, which seems reasonable pending comparison with the experimental results.

FUTURE ASPIRATIONS:

Further Analysis on model as well as comparison with data is needed.