2nd order LODE

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March 2022

Abstract

This appendix is related to solving ordinary diffential equations with a mathematically rigorous style.

1 Introduction

Consider the homogeneous equation:

$$\ddot{y} + \alpha\dot{y} + \beta y = 0 \tag{1}$$

By basic differentiation techniques, we have

$$\frac{d}{dx}(e^x) = e^x \tag{2}$$

$$\implies \frac{d}{dx}(ae^x) = ae^x \tag{3}$$

Suppose that f: $\mathbb{R} \to \mathbb{R}$ is differentiable everywhere and $f'(x) = af(x)$. There are no other functions in which the derivative is equal to itself.

1.1 Proof 1

If we let $g(x) = \frac{af(x)}{e^{ax}}$

$f(x)$ and e^{ax} are differentiable $\to g'(x)$ is defined.

$$g'(x) = \frac{f'(x)e^{ax} - af(x)e^{ax}}{(e^{ax})^2} \tag{4}$$

$$g'(x) = \frac{af(x)e^{ax} - af(x)e^{ax}}{(e^{ax})^2} \tag{5}$$

$$g'(x) = 0 \tag{6}$$

By identity theorem, $g'(x) = 0 \to g(x) = c$, where $c \in \mathbb{R}$
$\implies f(x) = Ae^{ax}$, where $c = Aa$. We take $a = 1$ and hence $A = c$. The above statement can be proved by Picard–Lindelöf theorem, however we won't cover it here.

1.2 Proof 2

Suppose that f: $\mathbb{R} \to \mathbb{R}$ is differentiable everywhere and $f(x) = f'(x)$

$$1 = \frac{f(x)}{f'(x)} \tag{7}$$

$$x + c = \int \frac{f(x)}{f'(x)}, c \in \mathbb{R} \tag{8}$$

$$x + c = ln(f(x)) \tag{9}$$

$$f(x) = e^{x+c} \tag{10}$$

$$f(x) = \gamma e^x, \gamma = e^c \tag{11}$$

1.3 Ansatz

Therefore, we take the Ansatz: $y = e^{\lambda x}$ and obtain:

$$\ddot{y} + \alpha\dot{y} + \beta y = 0 \tag{12}$$

$$(\lambda^2 + \alpha\lambda + \beta)(e^{\lambda x}) = 0 \tag{13}$$

Obviously $e^{\lambda x} = 0$ is undefined, so

$$\lambda^2 + \alpha\lambda + \beta = 0 \tag{14}$$

Note that the equation above is called the auxillary (characteristic) equation.

$$\lambda_1 = \frac{a}{2} + \sqrt{\frac{a^2}{4} - b} \tag{15}$$

$$\lambda_2 = \frac{a}{2} - \sqrt{\frac{a^2}{4} - b} \tag{16}$$

We can imagine both λ_1 and λ_2 satisfies the equation, so y is a linear combination of $e^{\lambda_1 x}$ and $e^{\lambda_2 x}$, we can rewrite this as:

$$y = Ae^{\lambda_1 x} + Be^{\lambda_1 x} \tag{17}$$

Where A & B $\in \mathbb{R}$

A Rigorous Statement: Let S be the solution space of

$$\ddot{y} + \alpha\dot{y} + \beta y = 0$$

If $\lambda_1 \neq \lambda_2$, then $\{e^{\lambda_1 x}, e^{\lambda_2 x}\}$ is a basis for S.

Note that the above is only true for $\lambda_1 \neq \lambda_2$. For $\lambda_1 = \lambda_2$,$\{e^{\lambda_1 x}, te^{\lambda_1 x}\}$ is a basis for S. The proof will be written some other time.

2 Substitution

2.1 Motivation

If we want to solve a non-homogeneous 2nd order differential equation, say

$$y" + ky + c = 0 \qquad \text{where } c \in \mathbb{R}$$

Getting rid of the constant would allow us to solve using the equation (17).

2.2 Construction

Let

$$u = y + \frac{c}{k} \tag{18}$$

By basic diffentiation techniques, we can obtain the first and second derivative:

$$u' = y' \tag{19}$$
$$u" = y" \tag{20}$$

So the equation becomes

$$u" + k(u - \frac{c}{k}) + c = 0 \tag{21}$$
$$u" + ku = 0 \tag{22}$$

From equation(14), we can see that the auxillary equation becomes:

$$\lambda^2 + k = 0 \tag{23}$$
$$(\lambda + \sqrt{k}i)(\lambda - \sqrt{k}i) = 0 \tag{24}$$

So:

$$\lambda_1 = \sqrt{k}i \tag{25}$$
$$\lambda_2 = -\sqrt{k}i \tag{26}$$

Therefore, the General Solution:[1]

$$u = Ae^{i\sqrt{k}x} + Be^{-i\sqrt{k}x} \tag{27}$$

However, this is not our final answer. For y:

$$y = u - \frac{c}{k} \tag{28}$$
$$y = Ae^{i\sqrt{k}x} + Be^{-i\sqrt{k}x} - \frac{c}{k} \tag{29}$$

[1]This equation reminds me of quantum mechanics, in particular, the Schrödinger equation. I may write a paper on this later.