I've Got a Three-Body Problem

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The Newtonian n-body problem: Suppose n bodies move in space under their mutual gravitational attraction. Given their initial positions and velocities, what is their future motion?

$$
\begin{aligned}
m_i &= \text{mass of the } i\text{-th body} \\
\mathbf{q}_i &= \text{position of the } i\text{-th body in } \mathbb{R}^2 \text{ or } \mathbb{R}^3 \\
r_{ij} &= ||\mathbf{q}_i - \mathbf{q}_j||
\end{aligned}
$$

Using Newton's inverse square law, the force on the i-th body from the j-th body is given by:

$$
\frac{m_i m_j}{||\mathbf{q}_i - \mathbf{q}_j||^2} \cdot \frac{\mathbf{q}_j - \mathbf{q}_i}{||\mathbf{q}_j - \mathbf{q}_i||}
$$

Using $F = ma$, the differential equation that determines the motion of the i-th body is:

$$
m_i \frac{d^2 \mathbf{q}_i}{dt^2} = \sum_{j \neq i}^{n} \frac{m_i m_j (\mathbf{q}_j - \mathbf{q}_i)}{||\mathbf{q}_i - \mathbf{q}_j||^3}
$$

Differential Equations

Goal: Find a function(s) (not a number!) which satisfies an equation(s) involving some unknown function(s) and its derivative(s).

1. $\frac{dy}{dt} = y$

One Solution: $y(t) = e^t$

Are there others?

Yes! $y(t) = 2e^t$ also satisfies the equation.

General Solution: $y(t) = y_0 e^t, \quad y_0 \in \mathbb{R}$

Here, $y_0 = y(0)$ is an initial condition (eg. money in your bank account, population of rabbits, etc.)

2. $\frac{d^2y}{dt^2} = 0$

One Solution: $y(t) = 1$

Another Solution: $y(t) = t$

General Solution: $y(t) = v_0 t + y_0$ where $y_0, v_0 \in \mathbb{R}$.
Note: $y_0 = y(0)$ and $v_0 = y'(0)$

3. $\frac{d^2y}{dt^2} = -y$

One Solution: $y(t) = \sin t$

Another Solution: $y(t) = \cos t$

General Solution: $y(t) = y_0 \cos t + v_0 \sin t$ where $y_0, v_0 \in \mathbb{R}$.
Note: $y_0 = y(0)$ and $v_0 = y'(0)$

The Kepler Problem

The case $n = 2$ can be reduced to a central force problem wherein one body (the largest), is assumed to be fixed and the other body orbits around it. This is commonly referred to as a Kepler problem.

The orbiting body will travel on a conic section (circle, ellipse, hyperbola, parabola or line). Newton essentially invented Calculus to prove this fact. It was initially discovered by Kepler (1571-1630) who based his famous three laws on data (stolen?) from Tycho Brahe (1546-1601).

The Planar Three-Body Problem

$$
\begin{aligned}
m_1 \ddot{x}_1 &= \frac{m_1 m_2 (x_2 - x_1)}{||\mathbf{q}_2 - \mathbf{q}_1||^3} + \frac{m_1 m_3 (x_3 - x_1)}{||\mathbf{q}_3 - \mathbf{q}_1||^3} \\
m_1 \ddot{y}_1 &= \frac{m_1 m_2 (y_2 - y_1)}{||\mathbf{q}_2 - \mathbf{q}_1||^3} + \frac{m_1 m_3 (y_3 - y_1)}{||\mathbf{q}_3 - \mathbf{q}_1||^3} \\
m_2 \ddot{x}_2 &= \frac{m_1 m_2 (x_1 - x_2)}{||\mathbf{q}_2 - \mathbf{q}_1||^3} + \frac{m_2 m_3 (x_3 - x_2)}{||\mathbf{q}_3 - \mathbf{q}_2||^3} \\
m_2 \ddot{y}_2 &= \frac{m_1 m_2 (y_1 - y_2)}{||\mathbf{q}_2 - \mathbf{q}_1||^3} + \frac{m_2 m_3 (y_3 - y_2)}{||\mathbf{q}_3 - \mathbf{q}_2||^3} \\
m_3 \ddot{x}_3 &= \frac{m_1 m_3 (x_1 - x_3)}{||\mathbf{q}_3 - \mathbf{q}_1||^3} + \frac{m_2 m_3 (x_2 - x_3)}{||\mathbf{q}_3 - \mathbf{q}_2||^3} \\
m_3 \ddot{y}_3 &= \frac{m_1 m_3 (y_1 - y_3)}{||\mathbf{q}_3 - \mathbf{q}_1||^3} + \frac{m_2 m_3 (y_2 - y_3)}{||\mathbf{q}_3 - \mathbf{q}_2||^3}
\end{aligned}
$$

Integrals of Motion

$$\begin{aligned} m_1\ddot{x}_1 + m_2\ddot{x}_2 + m_3\ddot{x}_3 &= 0 \\ m_1\ddot{y}_1 + m_2\ddot{y}_2 + m_3\ddot{y}_3 &= 0 \end{aligned}$$

$$\begin{aligned} m_1x_1(t) + m_2x_2(t) + m_3x_3(t) &= a_1t + b_1 \\ m_1y_1(t) + m_2y_2(t) + m_3y_3(t) &= a_2t + b_2 \end{aligned}$$

The integration constants $a_1, a_2, b_1, b_2 \in \mathbb{R}$ are called integrals or conserved quantities and are completely determined by the initial conditions.

For the general n-body problem:

$$\sum_{i=1}^{n} m_i\mathbf{q}_i(t) \;=\; \mathbf{a}t + \mathbf{b}$$

The center of mass moves uniformly along a line.

Other Conserved Quantities

$$\sum_{i=1}^{n} \mathbf{q}_i(t) \times m_i \dot{\mathbf{q}}_i(t) \;=\; \Omega$$

Angular momentum Ω is conserved. In the spatial n-body problem, this gives another three integrals of motion. In the planar problem, this gives one conserved quantity.

The conservation of angular momentum can be derived by differentiating the left-hand side with respect to t and obtaining 0.

Define the momentum as $\mathbf{p}_i = m_i \dot{\mathbf{q}}_i$. Let

$$K = \frac{1}{2} \sum_{i=1}^{n} m_i ||\dot{\mathbf{q}}_i||^2$$

$$U = \sum_{i<j} \frac{m_i m_j}{||\mathbf{q}_i - \mathbf{q}_j||}$$

K is the kinetic energy and U is the Newtonian potential.
Equations of motion:

$$\dot{\mathbf{q}}_i = \frac{\mathbf{p}_i}{m_i} = \frac{\partial H}{\partial \mathbf{p}_i}$$

$$\dot{\mathbf{p}}_i = \frac{\partial U}{\partial \mathbf{q}_i} = -\frac{\partial H}{\partial \mathbf{q}_i}$$

where $H = K - U$ is called the Hamiltonian. Since H is also conserved throughout the motion, this gives 6 constants of motion in the planar n-body problem and 10 in the spatial problem.

Solutions with Singularities

Equations of motion:

$$m_i \frac{d^2\mathbf{q}_i}{dt^2} = \sum_{j \neq i}^{n} \frac{m_i m_j (\mathbf{q}_j - \mathbf{q}_i)}{||\mathbf{q}_i - \mathbf{q}_j||^3}$$

What happens if, for some time $t = t^*$, $||\mathbf{q}_i(t^*) - \mathbf{q}_j(t^*)|| = 0$?

COLLISION!

Singularities

Recall: $r_{ij} = ||\mathbf{q}_i - \mathbf{q}_j||$ (distance between i-th and j-th bodies)

Let $r(t) = \min\{r_{ij}(t) : i,j \in \{1,\ldots,n\}, i \neq j\}$.

Definition

If $\lim_{t \to t^*} r(t) = 0$, then we say a singularity occurs at $t = t^*$.

Important observation: A singularity can result from a collision or a non-collision. In other words, if $r(t) \to 0$, it does not necessarily follow that $r_{ij}(t) \to 0$ for some pair i, j.

Painlevé (1895) showed that the only singularities in the three-body problem are due to collisions.

Jeff Xia (1992, Annals of Mathematics) showed that non-collision singularities exist in the 5-body problem.

A Good Guess: Central Configurations

Definition

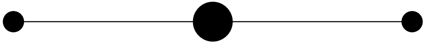
A *central configuration* (c.c.) is an initial configuration of bodies such that the acceleration vector for each body is proportional to its position vector. Specifically, for each index i,

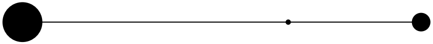
$$\sum_{j \neq i}^{n} \frac{m_i m_j (\mathbf{q}_j - \mathbf{q}_i)}{||\mathbf{q}_j - \mathbf{q}_i||^3} = \lambda m_i \mathbf{q}_i$$

for some proportionality constant λ.

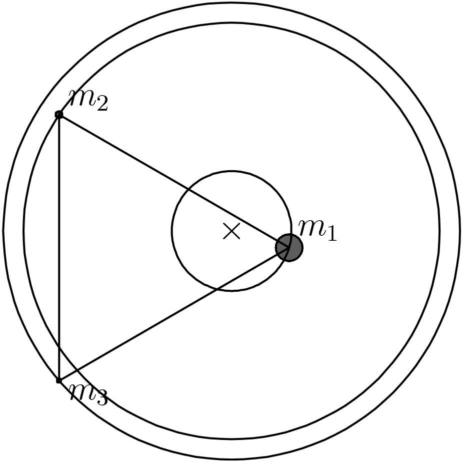
- Released from rest, a c.c. maintains the same shape as it heads toward total collision.
- Given the correct initial velocities, a c.c. will rigidly rotate about its center of mass. Such a solution is called a relative equilibrium.
- Any Kepler orbit can be attached to a c.c. to obtain a new solution to the n-body problem.

3-Body Collinear Configurations (Euler 1767)

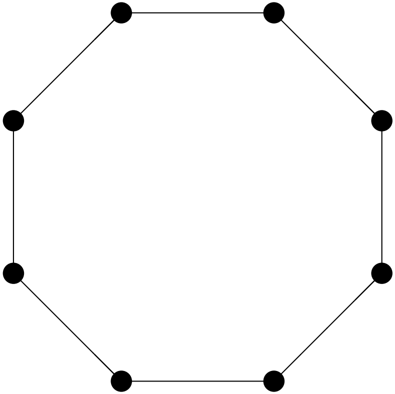




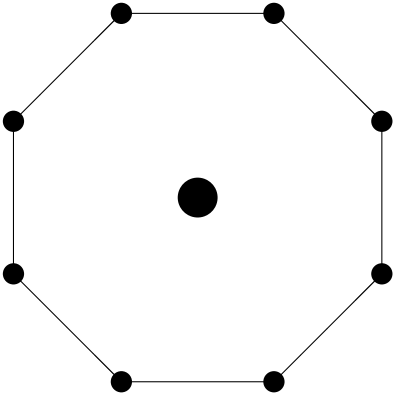
Equilateral Triangle (Lagrange 1772)



Regular n-gon (equal mass required for $n \geq 4$)



$1 + n$-gon (arbitrary central mass)



Used by Sir James Clerk Maxwell in 1859 in Stability of the Motion of Saturn's Rings (winner of the Adams Prize)

The Calculus of Variations

Goal: Find special planar periodic solutions using Hamilton's principle of least action. (Calculus on a space of curves.)

Let $\Sigma = \{\mathbf{q} \in \mathbb{R}^{2n} : \mathbf{q}_i = \mathbf{q}_j \text{ for some } i \neq j\}$ (collision set). The configuration space for the n-body problem is $\mathbb{R}^{2n} - \Sigma$. Let Γ_T denote the space of all absolutely continuous loops of period T in $\mathbb{R}^{2n} - \Sigma$.

The action of a path $\gamma \in \Gamma_T$ is

$$A(\gamma) = \int_0^T K(\dot{\gamma}(t)) + U(\gamma(t))\, dt$$

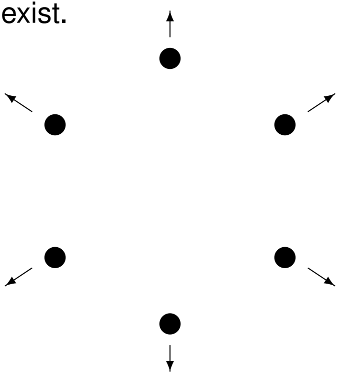
Amazing Fact: The principle of least action states that critical points (extremals) of the action are solutions to the n-body problem!

Since $K \geq 0$ and $U > 0$, $A(\gamma) > 0$. Moreover, as $\gamma \to \Sigma$, $U(\gamma) \to \infty$. Therefore, we typically seek to **minimize** the action.

Problems Using Variational Methods

The configuration space $\mathbb{R}^{2n} - \Sigma$ is not compact.

1. Minima might not exist.



2. A minimizing trajectory may contain collisions. This occurs in the Kepler problem where the minimizing solution is independent of the eccentricity. The ejection/collision solution is an action minimizer (Gordon 1970).

Theorem

(Chenciner, Montgomery 2000) There exists a figure-eight shaped curve $\mathbf{q} : (\mathbb{R}/T\mathbb{Z}) \mapsto \mathbb{R}^2$ *such that*

1. $\mathbf{q}(t) + \mathbf{q}(t + T/3) + \mathbf{q}(t + 2T/3) = 0 \;\; \forall t$
   (center of mass is at the origin.)
2. *Symmetry*

$$\mathbf{q}(t + T/2) = -\bar{\mathbf{q}}(t), \;\; \mathbf{q}(-t + T/2) = \bar{\mathbf{q}}(t) \;\; \forall t$$

3. $(\mathbf{q}(t + 2T/3), \mathbf{q}(t + T/3), \mathbf{q}(t))$ *is a zero angular momentum, periodic solution to the planar 3-body problem with equal masses.*

A solution where the n bodies follow each other along a single closed curve with equal phase shift is called a choreography.

Proof Outline:

Construct the orbit on the shape sphere, the space of oriented triangles.

1. Search for minimizers over the class of paths Γ traveling from an Euler central configuration (with say $\mathbf{q}_3$ at the center) to an isosceles configuration (with say $r_{12} = r_{13}$). This path will be $1/12$th of the full periodic orbit.
2. Let A_c be the smallest possible action for a path with collisions in Γ (compute via Kepler problem). Choose a simple test path (constant speed and potential) and compute its action A (numerically). Collisions are excluded by showing $A < A_c$.
3. The boundary terms of the first variation (integration by parts) and the symmetries induced by equality of masses allows for eleven copies of the minimizer to be fit together to create the full orbit.
4. A special area formula is used to reconstruct the motion in the phase space. By showing that the angular momentum of any one of the bodies vanishes only as the body passes through the origin, this implies the curve is a figure eight.

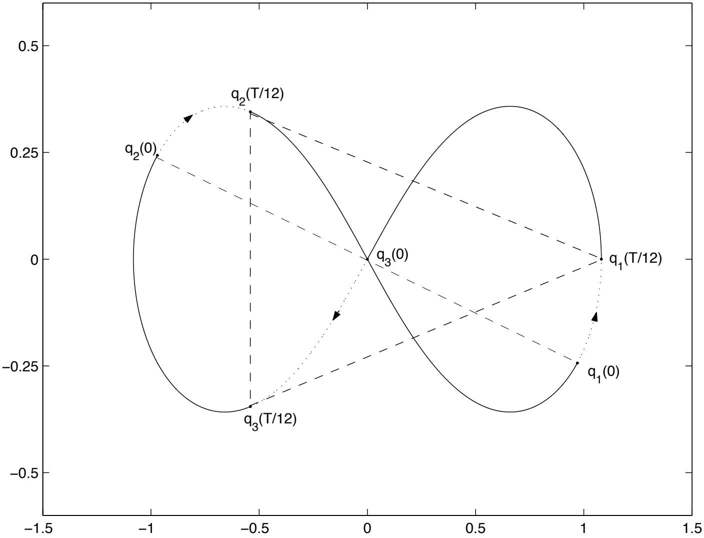


Figure: The first 12th of the figure-eight orbit (dotted), traveling from an Euler collinear central configuration to an isosceles triangle.

Note: The figure-eight orbit is stable! This is quite a surprise.

- The regular n-gon circular choreographies (equal mass) are all unstable.
- The Lagrange equilateral triangle solution is linearly stable only when one mass dominates the others.

$$\frac{m_1 m_2 + m_1 m_3 + m_2 m_3}{(m_1 + m_2 + m_3)^2} < \frac{1}{27}.$$

- If $n \geq 24,306$, all equal mass relative equilibria are unstable.
- The $1 + n$-gon relative equilibrium is linearly stable iff the central mass is at least $0.435 n^3$.
- All other known choreographies appear to be unstable.

On the other hand: Adding eccentricity to an unstable relative equilibrium could make it linearly stable (eg. Lagrange equilateral triangle)

Some Other Remarks About the Figure-eight

- Numerical experiments suggest that the probability of finding a figure-eight is somewhere between one per galaxy and one per universe.
- While slight variations in the initial positions or velocities of the figure-eight orbit do not alter the general shape of the eight, varying the masses even by a tiny amount destroys the orbit.
- The discovery of the figure-eight orbit and the variational techniques used to prove its existence led to many, many more wonderful orbits in the planar and spatial n-body problem. These include "hip-hop" orbits, intricate choreographies, figure-eights with an odd number of bodies, highly symmetric orbits and solutions with unequal masses.

Some Other Choreographies

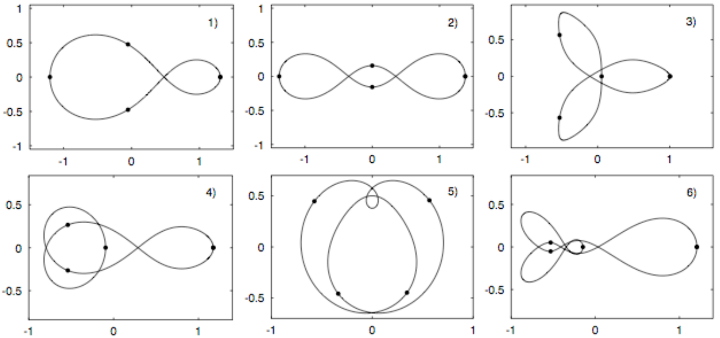
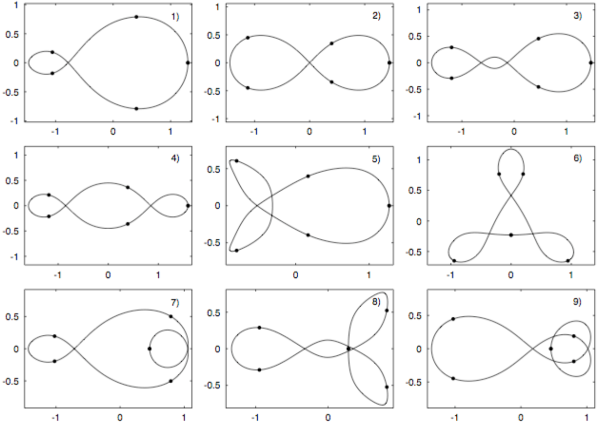


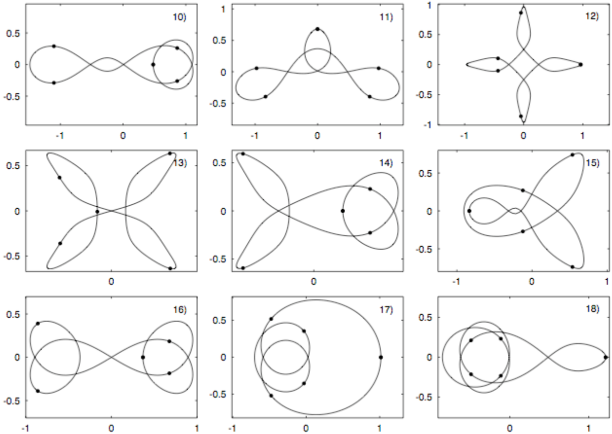
Figure 2. *Choreographies for four bodies under the Newtonian potential. Note that there is another choreography, also with $N = 4$, very similar to case number 5.*

Orbits numerically discovered by Carles Simó.

Some Choreographies for the 5-Body Problem (Simó)



More Choreographies for the 5-Body Problem (Simó)



Non-symmetric Choreographies (Simó)

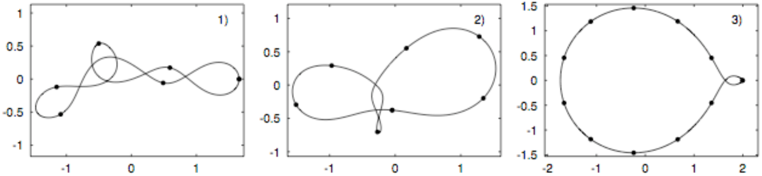


Figure 4. *Case 1: A choreography without any geometrical symmetry for* $N = 6$. *Case 2: Idem for* $N = 7$. *Case 3: A circle–like choreography with a small outer loop.*

The Planar, Circular, Restricted 3-Body Problem (PCR3BP)

Set up: Assume two large bodies (primaries) are traveling on circular orbits. Insert a third, infinitesimal body (satellite, comet, space ship) that has no influence on the two primaries. Change to a frame rotating with the same speed as the primaries.

$\mathbf{q}_1 = (1-\mu, 0), m_1 = \mu$ and $\mathbf{q}_2 = (-\mu, 0), m_2 = 1-\mu \quad (0 < \mu \leq 1/2)$

Let $a = \sqrt{(x-1+\mu)^2 + y^2}, \quad b = \sqrt{(x+\mu)^2 + y^2}$.

Equations of motion for the infinitesimal body (x, y):

$$\dot{x} = u \qquad \dot{y} = v$$
$$\dot{u} = V_x + 2v \qquad \dot{v} = V_y - 2u$$

where

$$V(x,y) = \frac{1}{2}(x^2+y^2) + \frac{\mu}{a} + \frac{1-\mu}{b} + \frac{1}{2}\mu(1-\mu)$$

is the amended potential.

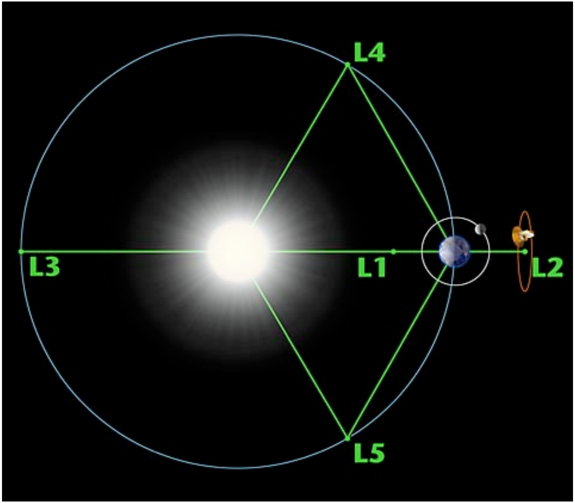


Figure: There are always five equilibria (libration points or Lagrange points) in the PCR3BP.

http://map.gsfc.nasa.gov/mission/observatory_l2.html

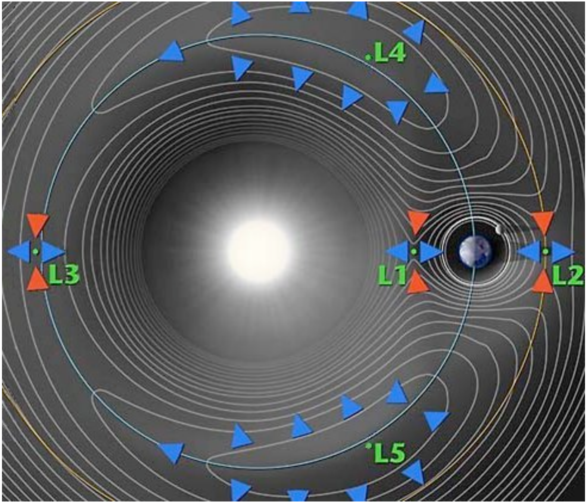


Figure: The level curves for the amended potential and the libration points.
http://map.gsfc.nasa.gov/mission/observatory_l2.html

Solutions Cannot Travel on Level Curves

The following theorem is a generalization of Saari's conjecture to the PCR3BP.

Theorem

(GR, Melanson 2007) The only solutions to the planar, circular, restricted three-body problem with a constant value of the amended potential V *are equilibria (libration points).*

Corollary

(GR, Melanson 2007) It is not possible for a solution to the PCR3BP to travel with constant speed without being fixed at one of the libration points.

Some Final Remarks

- Although the n-body problem is challenging, the field of celestial mechanics is full of interesting and accessible research problems that have important applications to spacecraft transport and understanding our universe. Some of these research problems are accessible to motivated undergraduates!
- A wide variety of branches of mathematics are used in the n-body problem: multivariable calculus, linear algebra, differential equations, analysis, dynamical systems, calculus of variations, geometry, topology, algebraic geometry
- Some of the greatest mathematicians have worked on the n-body problem: Newton, Euler, Lagrange, Poincaré, Smale, Hénon, Conley, ...
- Thank you for coming! Thank you organizers!